MATH2040 Linear Algebra II

Tutorial 1

September 15, 2016

1 Examples:

Example 1

Let V be a vector space over a field $\mathbb{F}$ of characteristic not equal to two, and let u and v be distinct vectors in V. Prove that $\{u, v\}$ is linearly independent if and only if $\{u+v, u-v\}$ is linearly independent.

Solution

We first recall that the characteristic of a field $\mathbb{F}$ is the smallest positive integer p such that $\overbrace{1+1+\cdots+1}^{p \text{ times}} = 0$, where 0 and 1 are the identity elements for addition and multiplication, respectively, in $\mathbb{F}$. Then, we could start our prove.

" $\Rightarrow$ " Suppose $\{u, v\}$ is linearly independent, then $au + bv = 0$ implies $a = b = 0$. Next, we assume that $c(u+v) + d(u-v) = 0$, and we want to show that $c = d = 0$.

Note, $(c+d)u + (c-d)v = 0$ implies $c + d = c - d = 0$, in other words, $c + c = d + d = 0$. Since $\mathbb{F}$ is of characteristic not equal to two, so we can conclude that $c = d = 0$.

" $\Leftarrow$ " Similar to the above arguments, suppose $\{u+v, u-v\}$ is linearly independent, then $a(u+v) + b(u-v) = 0$ implies $a = b = 0$. Again, we assume that $cu + dv = 0$, and we want to show that $c = d = 0$.

Since we can deduce that $\frac{c+d}{2}(u+v) + \frac{c-d}{2}(u-v) = cu + dv = 0$, so $\frac{c+d}{2} = \frac{c-d}{2} = 0$. And due to $\mathbb{F}$ is of characteristic not equal to two, therefore, we can conclude that $c = d = 0$.

Example 2

Let $\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \beta = \{1, x, x^2\}$.

(a) Define $\mathrm{T} : \mathrm{M}_{2\times 2}(\mathbb{R}) \to \mathrm{M}_{2\times 2}(\mathbb{R})$ by $\mathrm{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}$. Compute $[\mathrm{T}]_\alpha$.

(b) Define $\mathrm{T} : \mathrm{P}_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by $\mathrm{T}(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix}$, where $'$ denotes differentiation. Compute $[\mathrm{T}]_\beta^\alpha$.

(c) Define $\mathrm{T} : \mathrm{M}_{2\times 2}(\mathbb{R}) \to \mathrm{P}_2(\mathbb{R})$ by $\mathrm{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ax^2 + (b+c+d)x + 2d$. Compute $[\mathrm{T}]_\alpha^\beta$.

Solution

(a) $[\mathrm{T}]_\alpha = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

(b) $[\mathrm{T}]_\beta^\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

(c) $[\mathrm{T}]_\beta^\alpha = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

2 Exercises:

Question 1 (Section 1.5 Q13):

Let V be a vector space over a field of characteristic not equal to two, and let u, v, and w be distinct vectors in V. Prove that $\{u, v, w\}$ is linearly independent if and only if $\{u+v, u+w, v+w\}$ is linearly independent.

Question 2 (Section 2.2 Q3):

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let β be the standard ordered basis for $\mathbb{R}^2$ and $\gamma = \{(1, 1, 0), (0, 1, 1), (2, 2, 3)\}$. Compute $[T]_\beta^\gamma$. If $\alpha = \{(1, 2), (2, 3)\}$, compute $[T]_\alpha^\gamma$.

Question 3 (Section 4.2 Q7):

Let $A = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$. Evaluate the determinant of A by cofactor expansion along the second row.

Solution

(Please refer to the practice problem set 1.)